Rulesets for Beatty Games

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Abstract

We describe a ruleset for a 2-pile subtraction game with P-positions $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$ for any irrational $1 < \alpha < 2$, and β such that $1/\alpha + 1/\beta = 1$. We determine the α's for which the game can be represented as a finite modification of Generalized Wythoff and describe this modification.

1 Introduction

Generalized Wythoff (see [3]) is a two-player game played on two piles of tokens where each player can either (a) remove any positive amount of tokens from one pile or (b) remove $x > 0$ tokens from one pile and $y > 0$ from the other provided that $|x - y| < t$ where $t \geq 1$ is a parameter of the game. The player first unable to move loses (*normal* play).

The case $t = 1$, in which the second type of moves is to remove the same amount of tokens from both piles, is the classical *Wythoff* game [10], a modification of the game Nim. From among the extensive literature on Wythoff's game, we mention just three: [1], [3], [11].

We restrict attention to invariant subtraction games, such as Generalized Wythoff. An *invariant* subtraction game is a subtraction game in which every move can be made from every game position, provided only that every pile retains a nonnegative number of tokens after the move. Invariant vector games were defined formally in [5], and further explored in [2]. Furthermore,

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we assume that the piles are unordered. Additional references on invariant subtraction games are, for example, [7] and [9].

In every finite game, every position is either an N-position – a position from which the **N**ext player can win, or a P-position – a position from which the **P**revious player can win. Throughout the paper we consider normal play and thus $(0, 0)$ is always a P-position. It is known that the P-positions of Generalized Wythoff are $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$, where $\alpha = [1; t, t, t, \ldots]$ and β is such that $1/\alpha + 1/\beta = 1$.

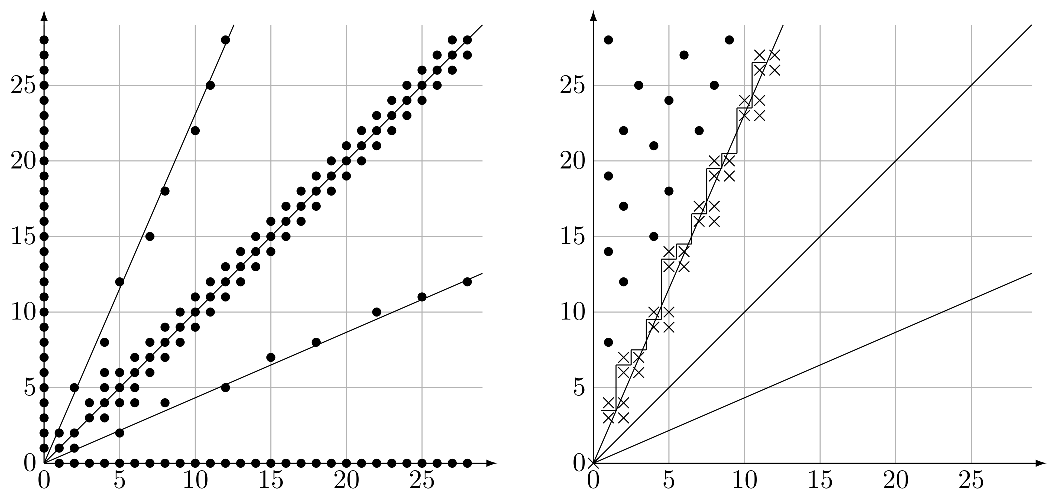
In [2] it was conjectured that for every irrational $1 < \alpha < 2$, the set $P_\alpha = \{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$ constitutes the set of P-positions of some invariant game. The conjecture was proven in [8]. We dub such games *Beatty games*. Note that even though the proof given in [8] is constructive, the ruleset is rather complicated, especially compared to the one of Generalized Wythoff. For special cases of α, simpler rulesets appear in the literature. For example, see [2] for a ruleset for the case $\alpha = [1; 1, q, 1, q, 1, q, \ldots]$ $(q \geq 1)$ or [9] for $\alpha = [1; q, 1, q, 1, \ldots]$ $(q \geq 1)$.

The aim of this paper is to suggest "compact" rulesets for all Beatty games. That is, for every irrational $1 < \alpha < 2$, find a compact ruleset whose corresponding P-positions are P_α. The term "compact" is a little vague. In this paper we give two different meanings for "compact": The first is a game whose moves are precisely those of Generalized Wythoff, except for some finite modification. We call such a game *GWM* (Generalized Wythoff Modified). We will prove the following theorem:

Theorem 1. *Let $1 < \alpha < 2$ be irrational. Then, there exists a GWM game whose P-positions are P_α if and only if*

$$\alpha^2 + b\alpha - c = 0 \quad \text{for some } b, c \in \mathbb{Z} \text{ such that } b - c + 1 < 0. \tag{1}$$

A consequence of this theorem is that for almost all α, *there is no* GWM ruleset. In fact, it will follow from the proof, that the N-positions of the form $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor - 1)$ require infinitely many new moves. This brings us to the second meaning of "compact": We show (see Theorem 2) that by adjoining the moves from these N-positions to $(0, 0)$ (together with finitely many additional moves) we obtain a ruleset for *every* irrational $1 < \alpha < 2$. Asymptotically, the moves of Generalized Wythoff are located on three lines: the x-axis, the y-axis and the $x = y$ diagonal. The moves described in Theorem 2 are located on 5 lines: the three lines mentioned above, together with the two lines: $\alpha x = \beta y$ and $\beta x = \alpha y$. This is illustrated in Figure 1(a). Hence



(a) A ruleset (b) The set $\mathscr{M}$ – forbidden subtractions

Figure 1: A Beatty game with $\alpha = [1; 2, 3, 4, \ldots]$

the second meaning we give to "a compact ruleset" is that asymptotically the moves are located on a finite number of lines (we also prefer to keep this number as small as possible).

This paper is structured as follows:

Section 2 describes the framework and introduces some notation.

In Section 3 we present the set $\mathscr{M}$ – the set of subtractions which connect one P-position to another. This set plays a crucial role in Theorem 1 and Theorem 2, as a move can be adjoined to the game if and only if it is not in $\mathscr{M}$. An example for the set $\mathscr{M}$, for $\alpha = [1; 2, 3, 4, \ldots]$, is shown in Figure 1(b).

In Section 4 we prove Theorem 2. We start with this theorem as it gives a more general result, and some of the techniques used to prove it are also used in the proof of Theorem 1.

Section 5 is dedicated to the proof of Theorem 1.

In Section 6 we present a detailed analysis for two special cases of GWM rulesets. As examples, we describe rulesets for $\alpha = \sqrt{12} - 2$ and for $\alpha = (\sqrt{301} - 15)/2$.

2 Preliminaries

A position in the game is denoted by a pair (X, Y) where X and Y are the sizes of the piles. A move, that allows a player to take $x \geq 0$ tokens from one pile and $y \geq 0$ tokens from the other is denoted by a pair (x, y). We use the convention that $X \leq Y$ and $x \leq y$. Note that, potentially, there can be two results of playing the move (x, y) from the position (X, Y): $(X - x, Y - y)$ and $(X - y, Y - x)$.

Let $\mathbb{V}$ denote the set of all possible subtraction moves:

$$\mathbb{V} = \{(x, y) \in \mathbb{Z}^2_{\geq 0} : x \leq y,\ 0 < y\}.$$

The ruleset of any invariant game (played on two unordered piles) is a subset of $\mathbb{V}$. For example, the ruleset of Generalized Wythoff is

$$GW(t) = \{(0, y) : y > 0\} \cup \{(x, y) : 0 < x \leq y \text{ and } y - x < t\} \subseteq \mathbb{V}.$$

The set $GW(0)$ is the ruleset of Nim, while $GW(1)$ is the ruleset of the classical Wythoff game.

In this paper, β always denotes $\alpha/(\alpha - 1)$ (so that $1/\alpha + 1/\beta = 1$). We demand $0 < \alpha < \beta$, which implies $1 < \alpha < 2 < \beta$.

For $x \in \mathbb{R}$, we write $x = \lfloor x \rfloor + \{x\}$ where $\lfloor x \rfloor \in \mathbb{Z}$ and $0 \leq \{x\} < 1$.

3 Forbidden subtractions

When suggesting a candidate for a ruleset $\mathcal{V} \subseteq \mathbb{V}$ whose P-positions should be $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$, one must check two things: (a) No P-position has a P-position follower and (b) Every N-position has a P-position follower. These two requirements are, in a sense, contrary: (a) bounds $\mathcal{V}$ from above while (b) bounds $\mathcal{V}$ from below.

This section deals with (a). In order to check whether (a) holds, construct the set $\mathscr{M} \subseteq \mathbb{V}$ of forbidden subtractions – those subtractions that connect one P-position to another. Then one simply checks that $\mathcal{V} \cap \mathscr{M} = \emptyset$. We have $\mathscr{M} = \mathscr{M}_1 \cup \mathscr{M}_2$ where $\mathscr{M}_1 = \{(\lfloor \alpha n \rfloor - \lfloor \alpha m \rfloor, \lfloor \beta n \rfloor - \lfloor \beta m \rfloor) : n > m \geq 0\}$ and $\mathscr{M}_2 = \{(\lfloor \alpha n \rfloor - \lfloor \beta m \rfloor, \lfloor \beta n \rfloor - \lfloor \alpha m \rfloor) : \lfloor \alpha n \rfloor > \lfloor \beta m \rfloor, m > 0\}$. See Figure 1(b) for an example of $\mathscr{M}$. The subtractions of $\mathscr{M}_1$ are represented as $\times$ while those of $\mathscr{M}_2$ are represented as $\bullet$.

Throughout this paper we will frequently use the following observation:

Observation 1. *Let $n, m, k \in \mathbb{Z}_{\geq 0}$ such that $n - m = k$. Then, $\lfloor \alpha n \rfloor - \lfloor \alpha m \rfloor = \lfloor \alpha k \rfloor + a$ where $a = 1$ if $\{\alpha n\} < \{\alpha k\}$, and $a = 0$ otherwise.*

In general, the structure of $\mathscr{M}_2$ is much more complicated than that of $\mathscr{M}_1$. See [4] for a detailed analysis of $\mathscr{M}_2$. Fortunately, this kind of detailed analysis is not necessary here. Instead, Proposition 1 below will suffice. We precede the proposition with the following geometric interpretation: the forbidden subtractions of $\mathscr{M}_2$ all lie above the line $\beta x = \alpha y$, see Figure 1(b). We will use this proposition to verify that the moves we adjoin to $\mathcal{V}$ are not in $\mathscr{M}_2$.

Proposition 1. *If $(\lfloor \alpha k \rfloor, y) \in \mathscr{M}_2$ then $y \geq \lfloor \beta k \rfloor + 1$. In addition, if $(\lfloor \alpha k \rfloor + 1, y) \in \mathscr{M}_2$ then $y \geq \lfloor \beta k \rfloor + 2$.*

Proof. Assume that $(\lfloor \alpha k \rfloor, y) \in \mathscr{M}_2$. Then there are $n > m > 0$ such that $\lfloor \alpha n \rfloor - \lfloor \beta m \rfloor = \lfloor \alpha k \rfloor$ and $\lfloor \beta n \rfloor - \lfloor \alpha m \rfloor = y$.

We have $\lfloor \beta m \rfloor = \lfloor \alpha n \rfloor - \lfloor \alpha k \rfloor \leq \lfloor \alpha(n-k) \rfloor + 1$. Therefore,

$$\begin{aligned} y &= \lfloor \beta n \rfloor - \lfloor \alpha m \rfloor = (\lfloor \beta n \rfloor - \lfloor \beta k \rfloor) - \lfloor \alpha m \rfloor + \lfloor \beta k \rfloor \geq \\ &\geq \lfloor \beta(n-k) \rfloor - \lfloor \alpha m \rfloor + \lfloor \beta k \rfloor \geq & \text{as } m, n-k > 0 \\ &\geq \lfloor \alpha(n-k) \rfloor - \lfloor \beta m \rfloor + 2 + \lfloor \beta k \rfloor \geq \lfloor \beta k \rfloor + 1. \end{aligned}$$

The second assertion is proven similarly. □

Now, consider the set $\mathscr{M}_1$. Note that one can write $\mathscr{M}_1 = \bigcup_{k=1}^{\infty} \mathscr{M}_1^k \cup P_\alpha$ where $\mathscr{M}_1^k := \{(\lfloor \alpha n \rfloor - \lfloor \alpha m \rfloor, \lfloor \beta n \rfloor - \lfloor \beta m \rfloor) : m > 0, n - m = k\}$. Fix $k \geq 1$ and consider the set $\mathscr{M}_1^k$. Write $x = \lfloor \alpha n \rfloor - \lfloor \alpha m \rfloor = \lfloor \alpha k \rfloor + a$ where $a \in \{0, 1\}$ (see Observation 1). Similarly, write $y = \lfloor \beta n \rfloor - \lfloor \beta m \rfloor = \lfloor \beta k \rfloor + b$.

Geometrically, the values of a and b are determined by the position of the point $(u, v) = (\{\alpha n\}, \{\beta n\})$ in $[0, 1)^2$ with respect to $p_k := (\{\alpha k\}, \{\beta k\})$.

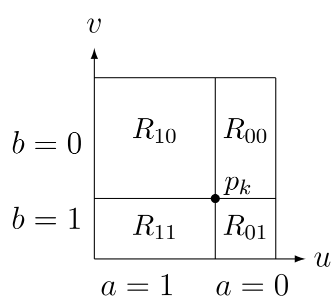


Figure 2: Determining a, b

Namely, divide $[0,1)^2$ into four open rectangles $R_{00}^{p_k}, R_{01}^{p_k}, R_{10}^{p_k}, R_{11}^{p_k}$ as shown in Figure 2. For example, $R_{11}^{p_k} = \{(u,v) : u < \{\alpha k\}, v < \{\beta k\}\}$. Then, $(\{\alpha n\}, \{\beta n\}) \in R_{ij}^{p_k}$ if and only if $a = i$ and $b = j$. The constraint $m > 0$ guarantees that $\{\alpha n\} \neq \{\alpha k\}$ and $\{\beta n\} \neq \{\beta k\}$.

The following proposition provides a criterion for testing whether the subtraction $(\lfloor \alpha k \rfloor + a, \lfloor \beta k \rfloor + b)$ is in $\mathscr{M}_1$. Let $D = \{(\{\alpha n\}, \{\beta n\}) : n \in \mathbb{Z}_{\geq 1}\} \subseteq [0,1)^2$ and let E be its topological closure.

Proposition 2. *Let $k \in \mathbb{Z}_{\geq 0}$ and let $a, b \in \{0,1\}$. Then, the subtraction $(\lfloor \alpha k \rfloor + a, \lfloor \beta k \rfloor + b)$ is in $\mathscr{M}_1$ if and only if either $a = b = 0$ or $E \cap R_{ab}^{p_k} \neq \emptyset$.*

Proof. In this proof we will omit the p_k from $R_{ab}^{p_k}$ and simply write R_{ab} instead. The case $a = b = 0$ is trivial so we assume otherwise. Assume that $E \cap R_{ab} \neq \emptyset$. Since R_{ab} is open, $D \cap R_{ab} \neq \emptyset$. Since D has no isolated points, $|D \cap R_{ab}| = \aleph_0$ and thus one can choose $(\{\alpha n\}, \{\beta n\}) \in D \cap R_{ab}$ with $n > k$. Choosing $m = n - k$, we obtain the requested result. For the second direction note that if $(\lfloor \alpha k \rfloor + a, \lfloor \beta k \rfloor + b) \in \mathscr{M}_1$ then necessarily $(\lfloor \alpha k \rfloor + a, \lfloor \beta k \rfloor + b) \in \mathscr{M}_1^k$. The rest of the proof is identical. □

Therefore we have to study the set $D = \{(\{\alpha n\}, \{\beta n\}) : n \in \mathbb{Z}_{\geq 1}\} \subseteq [0,1)^2$. The structure of D (or more accurately, of its topological closure, E) depends on the solutions of the equation

$$A\alpha + B\beta + C = 0, \qquad \text{where } A, B, C \in \mathbb{Z}. \tag{2}$$

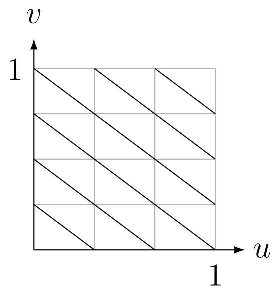
It is easy to see that the equation has a non-trivial solution if and only if α is the root of a quadratic polynomial with integer coefficients. In fact, if (A, B, C) is a solution then α will satisfy $A\alpha^2 + (B + C - A)\alpha - C = 0$. Note that we can choose A, B, C such that $\gcd(A, B, C) = 1$ and $A > 0$. These restrictions make the solution unique.

The following proposition is a result of Kronecker's theorem (see, for example, [6, ch. 23]).

Proposition 3. *If (2) has no non-trivial solution, then E is the entire $[0,1)^2$. Otherwise, $E = \{(u,v) \in [0,1)^2 : Au + Bv \in \mathbb{Z}\}$.*

An example for the set E, where $A = 3$ and $B = 4$ is shown in Figure 3.

As an example of how Proposition 3 and the above discussion may be used, we give here a short proof of Lemma 1 from [2]. The lemma states that for $\alpha = 1 + (\sqrt{t^2 + 4t} - t)/2$, and n such that $\lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor = 1$, we have



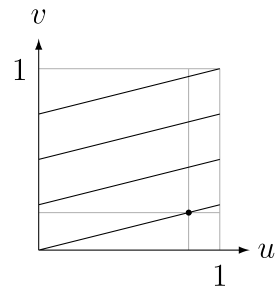


Figure 3: The set E for $A = 3$ and $B = 4$ Figure 4: Proof of Lemma 1 from [2]

$\lfloor\beta(n+1)\rfloor - \lfloor\beta n\rfloor = 2$. The α of Lemma 1 satisfies $1 \cdot \alpha - t \cdot \beta + (2t-1) = 0$. Proposition 3 implies that the points of D all lie on t segments, as shown in Figure 4. Moreover, one can easily check that the point $p_1 = (\{\alpha\}, \{\beta\})$ lies on the bottom segment. Recall that $\lfloor\alpha(n+1)\rfloor - \lfloor\alpha n\rfloor = \lfloor\alpha\rfloor + a = 1 + a$ and $\lfloor\beta(n+1)\rfloor - \lfloor\beta n\rfloor = \lfloor\beta\rfloor + b = 2 + b$. Here $a = 0$ and since $R_{01}^{p_1} \cap D = \emptyset$, we must have $b = 0$. This completes the proof.

4 A ruleset for an arbitrary α

Let $1 < \alpha < 2$ be an arbitrary irrational number. In this section we construct a game with a rather simple "one-line" ruleset for which the set of P-positions is P_α. An illustration for such a "one-line" ruleset is given in Example 1 on page 9.

We will construct the set of moves, $\mathcal{V}_\alpha$, gradually. As we adjoin moves to the game we must verify that they are not in $\mathscr{M}$ – this will guarantee that no P-position has a P-position follower. Moreover, we will have to adjoin enough moves such that every N-position will have a P-position follower.

The description of the rulesets we suggest (for an arbitrary α) appears in Theorem 2, which is presented in two parts: Theorem 2(a) deals with the case $\beta > 3$, and Theorem 2(b) deals with the case $2 < \beta < 3$.

4.1 $\beta > 4$

For the sake of simplicity, we assume first that $\beta > 4$. Denote $t = \lfloor\beta\rfloor - 1$. Partition the N-positions as follows: N_1 is the set of N-positions of the form $(\lfloor\alpha n\rfloor, \lfloor\beta n\rfloor - 1)$, N_2 is the set of N-positions of the form $(\lfloor\alpha n\rfloor + 1, \lfloor\beta n\rfloor - 1)$, and N_3 is the set of all other N-positions.

Lemma 1. *The ruleset $GW(t) \setminus \{(2, \lfloor\beta\rfloor)\}$ does not intersect $\mathscr{M}$ and allows the players to move from any position in N_3 to a P-position.*

Proof. Propositions 1 and 2 imply that the only move of $GW(t)$ which might be in $\mathscr{M}$ is $(2, \lfloor\beta\rfloor)$ so this move is excluded.

Let (x, y) be an N_3-position $(x \leq y)$. Let n be the maximal integer for which $y - x = \lfloor\beta n\rfloor - \lfloor\alpha n\rfloor + m$ for some $m \geq 0$. As the difference $(\lfloor\beta(n+1)\rfloor - \lfloor\alpha(n+1)\rfloor) - (\lfloor\beta n\rfloor - \lfloor\alpha n\rfloor)$ is at most $t+1$, we have $m \leq t$.

If $x \leq \lfloor\alpha n\rfloor$ then either $x = \lfloor\alpha k\rfloor$ or $x = \lfloor\beta k\rfloor$ for some $k \in \mathbb{Z}_{\geq 0}$. In both cases one can move to $(\lfloor\alpha k\rfloor, \lfloor\beta k\rfloor)$ using a Nim move.

Assume now that $x > \lfloor\alpha n\rfloor$. Consider the move $(x - \lfloor\alpha n\rfloor, y - \lfloor\beta n\rfloor)$ from the N-position (x, y) to the P-position $(\lfloor\alpha n\rfloor, \lfloor\beta n\rfloor)$. Note that $(y - \lfloor\beta n\rfloor) - (x - \lfloor\alpha n\rfloor) = m$ and $0 \leq m \leq t$. So as long as $m \neq t$ and $(x - \lfloor\alpha n\rfloor, y - \lfloor\beta n\rfloor) \neq (2, \lfloor\beta\rfloor)$, this is a valid move.

If $m = t$ then we have $(x, y) = (\lfloor\alpha(n+1)\rfloor + j, \lfloor\beta(n+1)\rfloor - 1 + j)$ for $j \geq 1$ ($j = 0$ gives an N_1-position). Then one can move to $(\lfloor\alpha(n+1)\rfloor, \lfloor\beta(n+1)\rfloor)$ (note that the move in this case is $(j-1, j)$ and it is valid since $\beta > 4$).

The last case we have to consider is $(x - \lfloor\alpha n\rfloor, y - \lfloor\beta n\rfloor) = (2, \lfloor\beta\rfloor)$. There are three possibilities for $(\lfloor\alpha(n+1)\rfloor, \lfloor\beta(n+1)\rfloor)$: $(x-1, y)$, $(x, y+1)$ and $(x-1, y+1)$. The first is disposed of by a Nim move. In the second (x, y) is an N_1-position, and in the third it is an N_2-position. □

For N_1-positions, we simply adjoin the following moves to the game:

$$F_\alpha = \{(\lfloor\alpha n\rfloor, \lfloor\beta n\rfloor - 1) : n \in \mathbb{Z}_{\geq 1}\},$$

which allow the player to move directly to $(0, 0)$ (note that as $\beta > 3$, none of these moves is in $\mathscr{M}$).

For N_2-positions, we could adjoin the moves $\{(\lfloor\alpha n\rfloor + 1, \lfloor\beta n\rfloor - 1) : n \in \mathbb{Z}_{\geq 1}\}$ as we did with N_1, but it is possible to solve this by adjoining finitely many moves instead. Take $n_0 \geq 2$ such that $\{\alpha n_0\} > 1 - \{\alpha\}$, and adjoin the moves:

$$\{(\lfloor\alpha n\rfloor + 1, \lfloor\beta n\rfloor - 1) : 1 < n < n_0\} \cup$$
$$\cup \{(\lfloor\alpha n_0\rfloor + 2, \lfloor\beta n_0\rfloor), (\lfloor\alpha n_0\rfloor + 2, \lfloor\beta n_0\rfloor - 1)\}.$$

Consider the N_2-position $(x, y) = (\lfloor\alpha n\rfloor + 1, \lfloor\beta n\rfloor - 1)$. We may assume that $n \geq n_0$ (otherwise we move to $(0, 0)$). If $\{\alpha n\} > 1 - \{\alpha\}$ then $\lfloor\alpha(n+1)\rfloor = \lfloor\alpha n\rfloor + 2$. Therefore $x = \lfloor\beta k\rfloor$ for some k, and so (x, y) is solved by a Nim move. If $\{\alpha n\} < \{\alpha n_0\}$, then we can move to $(\lfloor\alpha(n - n_0)\rfloor, \lfloor\beta(n - n_0)\rfloor)$.

We proved, so far only for $\beta > 4$:

Theorem 2(a). For $\beta > 3$, there exists a finite set of moves, S_α, such that the P-positions of the game defined by

$$\mathcal{V}_\alpha = \big(GW(\lfloor\beta\rfloor - 1) \setminus \{(2, \lfloor\beta\rfloor)\}\big) \cup F_\alpha \cup S_\alpha$$

are $\{(\lfloor\alpha n\rfloor, \lfloor\beta n\rfloor) : n \in \mathbb{Z}_{\geq 0}\}$.

Proof. Choose n_0 as above and let

$$\begin{aligned} S_\alpha = \quad & \{(\lfloor\alpha n\rfloor + 1, \lfloor\beta n\rfloor - 1) : 1 < n < n_0\} \cup \\ & \cup \{(\lfloor\alpha n_0\rfloor + 2, \lfloor\beta n_0\rfloor), (\lfloor\alpha n_0\rfloor + 2, \lfloor\beta n_0\rfloor - 1)\}. \end{aligned}$$

□

4.2 $3 < \beta < 4$

Note that the only place we used the fact that $\beta > 4$ was to prove that the move $(j-1, j)$ was valid in the case $m = t$, $(x, y) = (\lfloor\alpha(n+1)\rfloor + j, \lfloor\beta(n+1)\rfloor - 1 + j)$. If $\lfloor\beta\rfloor = 3$ and $j = 3$, then $(j-1, j) = (2, 3) = (2, \lfloor\beta\rfloor)$ is not a valid move.

Note that in this case $(x, y) = (\lfloor\alpha n\rfloor + 4, \lfloor\beta n\rfloor + 6)$, so the move $(4, 6)$ takes care of this case. Note that we can choose $n_0 = 2$ as $1 - \{\alpha\} < 2/3 < \{2\alpha\}$ and then $(4, 6)$ is already in $\mathcal{V}_\alpha$.

Thus we proved that Theorem 2(a) is true also for $3 < \beta < 4$.

Example 1. Let $\alpha = [1; 2, 3, 4, \ldots] \approx 1.43313$ and $\beta = [3; 3, 4, 5, \ldots] \approx 3.30879$. We have $n_0 = 2$ and $S_\alpha = \{(4, 5), (4, 6)\}$. Therefore, the possible moves are:

(a) Remove $x > 0$ tokens from one pile.
(b) Remove x tokens from one pile and y tokens from the other where $|x - y| < 2$. The move $x = 2$, $y = 3$ is not allowed.
(c) Remove 4 tokens from one pile and 6 tokens from the other.
(d) Remove $\lfloor\alpha n\rfloor$ tokens from one pile and $\lfloor\beta n\rfloor - 1$ tokens from the other.

The ruleset is shown in Figure 1(a) in the introduction.

4.3 $2 < \beta < 3$

This case is slightly more complicated, since:

1. $GW(1)$ is not enough here and we need $GW(2)$, which in turn has much more subtractions that are in $\mathscr{M}$ and thus should be excluded (previously we had only one: $(2, \lfloor\beta\rfloor)$).

2. The P-positions are more dense in the $\lfloor \beta n \rfloor$ direction and we cannot adjoin all the moves of F_α (since $F_\alpha \cap \mathscr{M}_1 \neq \emptyset$).

Start with the ruleset $GW(2)$, and for now ignore the fact that adjoining some of them is illegal. Consider the N-position (x, y). Take the maximal n such that $y - x = \lfloor \beta n \rfloor - \lfloor \alpha n \rfloor + m$ for $m \geq 0$. We have $m \in \{0, 1\}$. As in Section 4.1, we may assume $x > \lfloor \alpha n \rfloor$. But then one can move to $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor)$.

Now we have to exclude the moves in $GW(2) \cap \mathscr{M}$. Note that this is a finite set of moves. In fact, we will exclude the larger set:

$$G_\alpha = \left\{ (\lfloor \alpha k \rfloor + z, \lfloor \beta k \rfloor + w) : \begin{array}{c} k \geq 1,\ z, w \in \{0,1\} \text{ and} \\ \lfloor \beta k \rfloor + w - \lfloor \alpha k \rfloor - z < 2 \end{array} \right\}.$$

We have to make sure that for each excluded move in G_α, there is an alternative move. Let (x, y), n, m be as before and suppose that $x = \lfloor \alpha n \rfloor + \lfloor \alpha k \rfloor + z$ and $y = \lfloor \beta n \rfloor + \lfloor \beta k \rfloor + w$ for $k \geq 1$ and $z, w \in \{0, 1\}$. Note that $(\lfloor \alpha(n+k) \rfloor, \lfloor \beta(n+k) \rfloor)$ is also of the form $(\lfloor \alpha n \rfloor + \lfloor \alpha k \rfloor + a, \lfloor \beta n \rfloor + \lfloor \beta k \rfloor + b)$ for $a, b \in \{0, 1\}$. Figure 5 shows the 8 possible relative positions of the N-position (x, y) and the P-position $(\lfloor \alpha(n+k) \rfloor, \lfloor \beta(n+k) \rfloor)$. Note that we can rule out (a), (d), (f), (g) and (h) since they all contradict the maximality of n. (b) is solved by the Nim move $(0, 1)$, so we are left with (c) and (e). Once again the N-positions that require special treatment are $N_1 \cup N_2$.

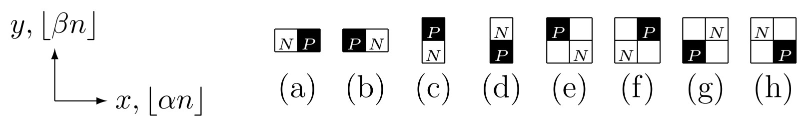


Figure 5: Relative positions of the
N-position and $(\lfloor \alpha(n+k) \rfloor, \lfloor \beta(n+k) \rfloor)$

We handle N_2 similarly to what we did in Section 4.1. Let n_0' be the smallest n such that $\lfloor \beta n \rfloor - \lfloor \alpha n \rfloor \geq 2$ (smaller n's do not correspond to N-positions). Find $n_0 \geq n_0'$ such that $\{\alpha n_0\} > 1 - \{\alpha\}$ and $\lfloor \beta n_0 \rfloor - \lfloor \alpha n_0 \rfloor \geq 3$. Then adjoin the moves:

$$\{(\lfloor \alpha n \rfloor + 1, \lfloor \beta n \rfloor - 1) : n_0' \leq n < n_0\} \cup$$

$$\cup \{(\lfloor \alpha n_0 \rfloor + 2, \lfloor \beta n_0 \rfloor), (\lfloor \alpha n_0 \rfloor + 2, \lfloor \beta n_0 \rfloor - 1)\}.$$

As we mentioned before, N_1 is slightly more complicated here, as we cannot simply adjoin all of F_α. Fortunately, we cannot adjoin $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor - 1)$

only when $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor - 1) = (\lfloor \alpha(n-1) \rfloor + 1, \lfloor \beta(n-1) \rfloor + 1)$ and in this case we can play the move $(1,1)$. Thus, we adjoin the moves: $F_\alpha \setminus (F_\alpha + (1,2))$ (where $F_\alpha + (1,2)$ is the set $\{(\lfloor \alpha n \rfloor + 1, \lfloor \beta n \rfloor + 1) : n \in \mathbb{Z}_{\geq 1}\}$).

The above discussion proves:

Theorem 2(b). For $2 < \beta < 3$, there exists a finite set of moves, S_α, such that the P-positions of the game defined by

$$\mathcal{V}_\alpha = (GW(2) \setminus G_\alpha) \cup (F_\alpha \setminus (F_\alpha + (1,2))) \cup S_\alpha$$

are $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$.

Proof. Choose n_0' and n_0 as above and let

$$\begin{aligned} S_\alpha = \quad & \{(\lfloor \alpha n \rfloor + 1, \lfloor \beta n \rfloor - 1) : n_0' \leq n < n_0\} \cup \\ & \cup \{(\lfloor \alpha n_0 \rfloor + 2, \lfloor \beta n_0 \rfloor), (\lfloor \alpha n_0 \rfloor + 2, \lfloor \beta n_0 \rfloor - 1)\}. \end{aligned}$$

□

5 Generalized Wythoff Modified (GWM)

A disadvantage of the description of the ruleset described in Section 4 is that it involves α explicitly. We can ask the following question: Can we describe a ruleset that doesn't involve α?

Of course, cardinality considerations imply that this is not possible for all α, as we have $\aleph$ different α's but only $\aleph_0$ finite descriptions of rulesets.

Therefore this will be possible only for a subset of the α's.

We start with two examples:

Example 2. Let $t \geq 1$. For $\alpha = [1; t, t, t, \ldots]$, the ruleset of Generalized Wythoff satisfies the requirement.

Example 3. In [2], the authors give the following set of moves for $\alpha = [1; 1, q, 1, q, \ldots]$: Wythoff moves ($GW(1)$) except for the moves: $(2,2)$, $(4,4)$, $\ldots$, $(2q-2, 2q-2)$; but with the move $(2q+1, 2q+2)$ adjoined.

Note that this representation has no "α-dependent" moves. Instead, it is a finite modification of Generalized Wythoff. In light of the last example we make the following definition:

Definition 1. (i) A ruleset $\mathcal{V}$ is said to be *GWM* (Generalized Wythoff Modified) if it is of the form $\mathcal{V} = GW(t) \triangle S$ where $\triangle$ denotes the symmetric difference, $S \subseteq \mathbb{V}$ is a finite subset of moves and $t \geq 1$.

(ii) Let $1 < \alpha < 2$ be irrational. We say that α is *GWM*, if there exists a GWM ruleset whose corresponding P-positions are $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$.

In this section we prove Theorem 1 that was stated in the introduction: α is GWM if and only if α satisfies (1) (see page 2). It is easy to see that (1) holds if and only if (2) (see page 6) has a solution with $A = 1$ and $B < 0$.

Proof of Theorem 1. We first prove that if α is GWM then (2) has a solution with $A = 1$ and $B < 0$. Let $\mathcal{V}$ be a GWM ruleset for P_α. First, consider the case where (2) has a solution with $A > 1$ and $B < 0$. Figure 6(a) shows the set $\{(\{\alpha n\}, \{\beta n\}) : n \in \mathbb{Z}_{\geq 0}\}$ in such a case. We focus on n's for which the point $(\{\alpha n\}, \{\beta n\})$ is very close to the point $(1/A, 0)$. Formally, take a sequence $\{n_i\}_{i=1}^{\infty}$ such that $\{\alpha n_i\} \to 1/A$ and $\{\beta n_i\} \to 0$ as $i \to \infty$.

For these n_i's, consider the N-position $(\lfloor \alpha n_i \rfloor, \lfloor \beta n_i \rfloor - 1)$. There must be a move in $\mathcal{V}$ to a P-position $(\lfloor \alpha m_i \rfloor, \lfloor \beta m_i \rfloor)$. Let $k_i = n_i - m_i \geq 1$. Note that there can be two moves that take $(\lfloor \alpha n_i \rfloor, \lfloor \beta n_i \rfloor - 1)$ to $(\lfloor \alpha m_i \rfloor, \lfloor \beta m_i \rfloor)$: $(\lfloor \alpha n_i \rfloor - \lfloor \alpha m_i \rfloor, \lfloor \beta n_i \rfloor - 1 - \lfloor \beta m_i \rfloor)$ and $(\lfloor \alpha n_i \rfloor - \lfloor \beta m_i \rfloor, \lfloor \beta n_i \rfloor - 1 - \lfloor \alpha m_i \rfloor)$.

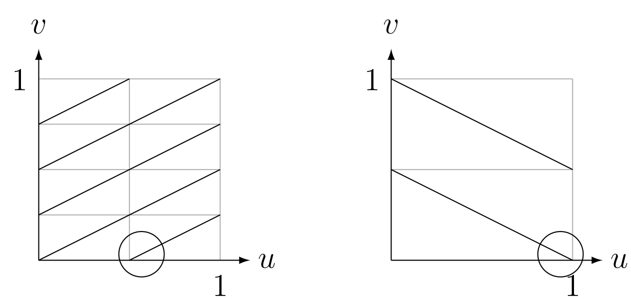
For the second type, we have

$$(\lfloor \beta n_i \rfloor - 1 - \lfloor \alpha m_i \rfloor) - (\lfloor \alpha n_i \rfloor - \lfloor \beta m_i \rfloor) \approx (\beta - \alpha)(n_i + m_i) \to \infty.$$

Hence this move can be in $\mathcal{V}$ only for finitely many n_i's.

For the first type, we have $(\lfloor \alpha n_i \rfloor - \lfloor \alpha m_i \rfloor, \lfloor \beta n_i \rfloor - 1 - \lfloor \beta m_i \rfloor) = (\lfloor \alpha k_i \rfloor + a, \lfloor \beta k_i \rfloor + b - 1)$ where $a, b \in \{0, 1\}$.

If $(a, b) = (0, 1)$ then this move is a P-position and therefore cannot be in $\mathcal{V}$.



(a) $A = 2$, $B = -4$ (b) $A = 1$, $B = 2$

Figure 6: Proof of Theorem 1($\Rightarrow$)

Assume that $(a, b) = (1, 1)$. We have $\{\alpha n_i\} < \{\alpha k_i\}$. Since $(\{\alpha n_i\}, \{\beta n_i\})$ was chosen to be close to $(1/A, 0)$ we may assume $\{\alpha n_i\} > 1/A$. Hence $R_{10}^{p_{k_i}} \cap E \neq \emptyset$. Proposition 2 implies that $(\lfloor \alpha k_i \rfloor + 1, \lfloor \beta k_i \rfloor + 0)$ connects two P-positions, which means that this move cannot be in $\mathcal{V}$.

For the other two cases ($b = 0$) we must have $\{\beta n_i\} > \{\beta k_i\} > 0$ and this is impossible as $\{\beta n_i\} \to 0$ (as there can be only finitely many different k_i's).

This completes the proof for the case that (2) has a solution with $A > 1$, $B < 0$. We will now prove the remaining two cases: (a) (2) has a solution with $B > 0$ and (b) (2) has no non-trivial solution. In both cases, we can choose a sequence $\{n_i\}_{i=1}^{\infty}$ such that $(\{\alpha n_i\}, \{\beta n_i\}) \to (1, 0)$ as $i \to \infty$ (see, for example, Figure 6(b)).

We consider the N-position $(\lfloor \alpha n_i \rfloor, \lfloor \beta n_i \rfloor - 1)$. As in the first case, we only have to consider the moves: $(\lfloor \alpha n_i \rfloor - \lfloor \alpha m_i \rfloor, \lfloor \beta n_i \rfloor - 1 - \lfloor \beta m_i \rfloor) = (\lfloor \alpha k_i \rfloor + a, \lfloor \beta k_i \rfloor + b - 1)$. But, as $(\{\alpha n_i\}, \{\beta n_i\}) \to (1, 0)$, for all but finitely many n_i's, we have $\{\alpha n_i\} > \{\alpha k_i\}$ and $\{\beta n_i\} < \{\beta k_i\}$. For these n_i's we have $(a, b) = (0, 1)$, so the move is $(\lfloor \alpha k_i \rfloor, \lfloor \beta k_i \rfloor)$, but this is a P-position so this move cannot be in $\mathcal{V}$.

Second direction Assume that (2) has a solution with $A = 1$ and $B < 0$ and denote $k = -B$. We show how to construct a GWM ruleset for α.

We assume first that $\beta > 3$. Note that the set of moves given in Section 4.1 has only one component that is not a finite modification of $GW(\lfloor \beta \rfloor - 1)$: F_α. Let $\mathcal{V}'_\alpha$ be the set of moves suggested there, without adjoining F_α. The set $\mathcal{V}'_\alpha$ satisfies: (a) it is a finite modification of $GW(\lfloor \beta \rfloor - 1)$, (b) $\mathcal{V}'_\alpha \cap \mathscr{M} = \emptyset$ and (c) it allows the players to move from N_2- and N_3-positions to P-positions.

As in the proof we gave for Lemma 1 from [2] (see page 6), the set E consists of k segments as illustrated in Figure 7 (for $B = -4$).

To handle N_1-positions, find two points $p_r = (\{\alpha r\}, \{\beta r\})$ and $p_s =$

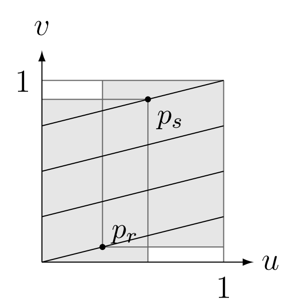


Figure 7: The set E with the points p_r and p_s

$(\{\alpha s\}, \{\beta s\})$ in D such that $D \subseteq R_{00}^{p_r} \cup R_{11}^{p_s}$ (see Figure 7) and adjoin the moves $(\lfloor \alpha r \rfloor, \lfloor \beta r \rfloor - 1)$, $(\lfloor \alpha s \rfloor + 1, \lfloor \beta s \rfloor)$. Note that since $R_{10}^{p_s} \cap D = \emptyset$, Proposition 2 implies that $(\lfloor \alpha s \rfloor + 1, \lfloor \beta s \rfloor) \notin \mathscr{M}_1$. We can use these moves to take care of $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor - 1)$ for $n \geq \max\{r, s\}$: if $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor - 1)$ is an N_1-position with $(\{\alpha n\}, \{\beta n\}) \in R_{00}^{p_r}$ then we can move to $(\lfloor \alpha(n-r) \rfloor, \lfloor \beta(n-r) \rfloor)$ and if $(\{\alpha n\}, \{\beta n\}) \in R_{11}^{p_s}$ then we can move to $(\lfloor \alpha(n-s) \rfloor, \lfloor \beta(n-s) \rfloor)$. Now adjoin a move from F_α for each $n < \max\{r, s\}$.

We now turn to the case $\beta < 3$. Here the only infinite component is $F_\alpha \setminus (F_\alpha + (1,2))$. We solve it by finding two points p_r, p_s as before, except now we have one additional restriction: the move that corresponds to p_r must not be in $\mathscr{M}$. This translates to $p_r \notin R_{00}^{p_1}$. It is easy to see that this additional requirement can also be satisfied. $\square$

6 Explicit rulesets

We can now analyze more carefully the finite modification whose existence is stated in Theorem 1($\Leftarrow$). We assume $A = 1$, $B < 0$ and denote $k = -B$. Fix $k > 1$ (for $k = 1$ we get Generalized Wythoff). The α's which satisfy (1) such that $-k = B = b - c + 1$ are now parametrized by $c \geq 2k - 1$. Figure 8 demonstrates how the points $(\{\alpha\}, \{\beta\})$ (where $k = 3$, $c \geq 5$) can be obtained by intersecting the curve induced by $1/\alpha + 1/\beta = 1$ and the k segments induced by $\{\alpha\} - k\{\beta\} \in \mathbb{Z}$, and shows how $c \pmod k$ determines the segment on which the point lies.

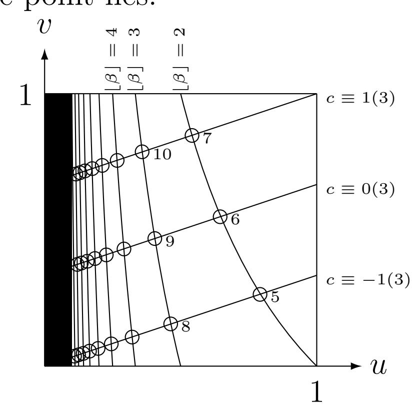


Figure 8: The point $(\{\alpha\}, \{\beta\})$ for $k = 3$ and different c's

Example 4. Consider the case $k = 3$ and $c = 5$. We have $b = 1$ and therefore $\alpha^2 + \alpha - 5 = 0$. Hence, $\alpha = (\sqrt{21} - 1)/2 = [1; 1, 3, 1, 3, \ldots] \approx 1.79129$ and $\beta = (\sqrt{21} + 9)/6 \approx 2.26376$. Indeed $\{\alpha\} - 3\{\beta\} = 0 \in \mathbb{Z}$ and $\{\beta\} < 1/3$.

It is easy to see that $\lfloor \beta \rfloor = \lfloor (c+1)/k \rfloor$. So we can write $c = k\lfloor \beta \rfloor + \tilde{c} - 1$ where $0 \leq \tilde{c} < k$. Table 1 shows the minimal values of r and s such that $D \subseteq R_{00}^{p_r} \cup R_{11}^{p_s}$ for $k = 3$ (see the proof of Theorem 1($\Leftarrow$)). It can be seen from the table that for c large enough, the values of r and s depend strongly on $\tilde{c}$:

$$r = \begin{cases} 1, & \tilde{c} = 0 \\ 3, & \tilde{c} = 1, 2 \end{cases}, \qquad s = \begin{cases} 2\lfloor \beta \rfloor, & \tilde{c} = 0 \\ 5, & \tilde{c} = 1 \\ 4, & \tilde{c} = 2 \end{cases}.$$

For small c's this analysis does not hold. We therefore focus only on c large enough, as for small c, each case can be investigated individually anyway.

Note that "large enough c" is equivalent to "small enough $\{\alpha\}$". In particular, we will assume that we deal with the case $\beta > 3$ ($\{\alpha\} < 1/2$).

In the rest of this section, we analyze two special cases: $\tilde{c} = 0$ and $\gcd(\tilde{c}, k) = 1$.

c	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\lfloor \beta \rfloor$	2	2	2	3	3	3	4	4	4	5	5	5	6	6	6
$\tilde{c}$	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
r	4	2	6	1	9	4	1	5	4	1	5	3	1	3	3
s	3	7	1	6	2	5	8	2	1	10	2	4	12	5	4
c	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
$\lfloor \beta \rfloor$	7	7	7	8	8	8	9	9	9	10	10	10	11	11	11
$\tilde{c}$	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
r	1	3	3	1	3	3	1	3	3	1	3	3	1	3	3
s	14	5	4	16	5	4	18	5	4	20	5	4	22	5	4

Table 1: The minimal values for r, s such that $D \subseteq R_{00}^{p_r} \cup R_{11}^{p_s}$ for $k = 3$

6.1 Case I: $\tilde{c} = 0$

Note that we have $\alpha = [1; t, tk, t, tk, \ldots]$ and $\beta = [1 + t; tk, t, tk, t, \ldots]$ where $t = \lfloor \beta \rfloor - 1$. This case can be thought of as a generalization of the case

$[1;1,k,1,k,\ldots]$ described in [2]. That being said, in this paper we only deal with the case $t \geq 2$.

In this case, the point $(\{\alpha\},\{\beta\})$ lies on the bottom segment. From the continued fraction of β we learn that the smallest i for which $\{\beta i\} > (k-1)/k$ (that is, the point is on the top segment) is $i = t(k-1)+1$. Choose $r = 1$ and $s = t(k-1)+2$. Since $t \geq 2$, $(\{\alpha s\},\{\beta s\})$ is on the top segment and $\{\alpha s\} \geq \{\alpha\}$ which guarantee that $D \subseteq R_{00}^{p_r} \cup R_{11}^{p_s}$. For an illustration, see Figure 9.

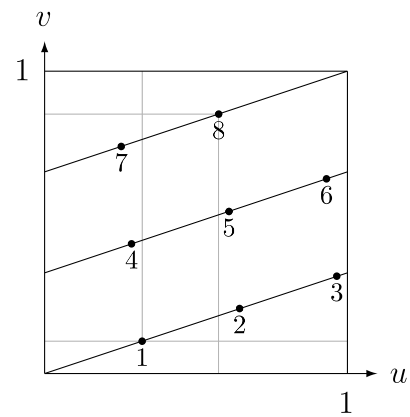


Figure 9: The points $(\{\alpha i\},\{\beta i\})$ for $1 \leq i \leq 8$, $k = 3$, $c = 11$

Let $i \leq s$. Observe that $\{\beta i\} \geq \{\beta\}$. It follows that $p_i \in R_{00}^{p_1} \cup R_{10}^{p_1}$ and thus one of the two moves $(\lfloor\alpha\rfloor, \lfloor\beta\rfloor - 1)$ and $(\lfloor\alpha\rfloor + 1, \lfloor\beta\rfloor - 1)$ takes care of the N_1-position $(\lfloor\alpha i\rfloor, \lfloor\beta i\rfloor - 1)$. Note that both moves are already contained in $GW(t)$, so it remains to adjoin the move $(\lfloor\alpha s\rfloor + 1, \lfloor\beta s\rfloor)$ to handle N_1-positions with $p_i \in R_{11}^{p_s}$.

It is easy to verify that

$$\lfloor\alpha s\rfloor = (t+1)(k-1)+2, \quad \lfloor\beta s\rfloor = (t+1)(t(k-1)+2).$$

Recall that in Theorem 2(a) we adjoined moves to deal with N_2-positions. Fortunately, in this special case, whenever $(x - \lfloor\alpha n\rfloor, y - \lfloor\beta n\rfloor) = (2, \lfloor\beta\rfloor)$ (see the proof of Lemma 1), we have $(x, y) \neq (\lfloor\alpha(n+1)\rfloor + 1, \lfloor\beta(n+1)\rfloor - 1)$. This is due to the fact that $R_{01}^{p_1} \cap D = \emptyset$. Thus, these additional moves are not necessary.

We proved the following:

Proposition 4. *Let* $\alpha = [1; t, tk, t, tk, \ldots]$ *for* $t > 1$ *and* $k > 1$. *The* P-*positions of the game defined by the moves*

$$GW(t) \setminus \big\{(2, t+1)\big\} \cup \big\{\big((t+1)(k-1)+3, (t+1)(t(k-1)+2)\big)\big\}$$

are P_α.

Example 5. Consider the ruleset of Generalized Wythoff with $t = 2$ where the move $(2, 3)$ is excluded but $(9, 18)$ is permitted. It follows from Proposition 4 that the P-positions are P_α for $\alpha = \sqrt{12} - 2 = [1; 2, 6, 2, 6, \ldots]$. Here $k = 3$ and $c = 8$.

6.2 Case II: $\tilde{c}$ and k are coprime

Note that as long as $i\{\alpha\} < 1$ (recall that we assumed that $\{\alpha\}$ is small enough), the point $(\{\alpha i\}, \{\beta i\})$ is on the $i \cdot \tilde{c} \pmod{k}$ segment (where 0 is the bottom segment). The smallest i on the top segment is therefore $1 \le d < k$ such that $d \equiv -\tilde{c}^{-1} \pmod{k}$. Choose s to be the second time it happens: $s = k + d$, and choose $r = k$ so that $r < s$ and $(\{\alpha r\}, \{\beta r\})$ is on the bottom segment.

We have,

$$\begin{aligned} \lfloor \alpha r \rfloor &= r = k, & \lfloor \beta r \rfloor &= k\lfloor \beta \rfloor + \tilde{c} = c + 1, \\ \lfloor \alpha s \rfloor &= s = k + d, & \lfloor \beta s \rfloor &= c + ((c+1)d + 1)/k. \end{aligned}$$

Proposition 5. *Suppose that* $c \ge 2k^2$ *and* $\gcd(c+1, k) = 1$. *Let* $1 \le d < k$ *be such that* $d \equiv -(1+c)^{-1} \pmod{k}$. *Then, the* P-*positions of the game defined by the moves*

$$\begin{aligned} & GW(\lfloor \beta \rfloor - 1) \setminus \big\{(2, \lfloor \beta \rfloor)\big\} \cup \\ \cup\ & \big\{(k+2, c), (k+2, c+1), \\ & (k+d+1, c + ((c+1)d+1)/k)\big\} \cup \\ \cup\ & \{(i+1, \lfloor i(c+1)/k \rfloor - 1) : 1 \le i \le k\} \cup \\ \cup\ & \{(i, \lfloor i(c+1)/k \rfloor - 1) : 1 \le i \le k\} \end{aligned}$$

are P_α.

Proof. Note that the fact that $c \ge 2k^2$ implies that $\{\alpha\} < 1/(2k-1)$ and therefore $i\{\alpha\} < 1$ for all $1 \le i \le k + d$. As explained above, in this case we can choose $r = k$ and $s = k + d$ (see Theorem 1($\Leftarrow$)).

The basic moves are $GW(\lfloor\beta\rfloor - 1) \setminus \{(2, \lfloor\beta\rfloor)\}$. As in the proofs of Theorem 1($\Leftarrow$) and Theorem 2(a) we have to deal with N_1- and N_2-positions.

Let $n \in \mathbb{Z}_{\geq 1}$ and consider the two positions $(\lfloor\alpha n\rfloor, \lfloor\beta n\rfloor - 1)$ and $(\lfloor\alpha n\rfloor + 1, \lfloor\beta n\rfloor - 1)$. If $n \leq k$ we can move directly to $(0, 0)$ for both positions.

For the N_1-position, we use the move $(\lfloor\alpha r\rfloor, \lfloor\beta r\rfloor - 1) = (k, c)$ or $(\lfloor\alpha r\rfloor + 1, \lfloor\beta r\rfloor - 1) = (k + 1, c)$ if $p_n \in R_{00}^{p_r}$ or $p_n \in R_{10}^{p_r}$ respectively. Otherwise $p_n \in R_{11}^{p_s}$ and we use the move $(\lfloor\alpha s\rfloor+1, \lfloor\beta s\rfloor) = (k+d+1, c+((c+1)d+1)/k)$.

For the N_2-position, we use the move $(k + 1, c)$ if $p_n \in R_{00}^{p_r}$, $(k + 2, c)$ if $p_n \in R_{10}^{p_r}$ and $(k + 2, c + 1)$ if $p_n \in R_{11}^{p_r}$. □

Example 6. For $k = 3$ and $c = 19$ we have $\alpha = (\sqrt{301} - 15)/2$, $\lfloor\beta\rfloor = 6$, $\tilde{c} = 2$ and $d = 1$. The ruleset that corresponds to P_α is the ruleset of Generalized Wythoff with $t = 5$ where the move $(2, 6)$ is excluded but the following moves are permitted:

$$(2, 12), (3, 12), (3, 19), (4, 19), (5, 19), (5, 20), (5, 26).$$

7 Conclusion

In this paper we presented "compact" rulesets for Beatty games. A Beatty game is an invariant subtraction game, played on two unordered piles of tokens, whose P-positions are $P_\alpha = \{(\lfloor\alpha n\rfloor, \lfloor\beta n\rfloor) : n \in \mathbb{Z}_{\geq 0}\}$ for some irrational $1 < \alpha < 2$ and β such that $1/\alpha + 1/\beta = 1$.

Theorem 1 shows that the α's for which there exists a finite modification of Generalized Wythoff (for some $t \geq 1$) are exactly the algebraic integers of degree 2 with one constraint – the minimal polynomial must satisfy $f(1) < 0$. The theorem also explains how to construct such a ruleset for an α with this property. For some special cases, we explicitly described the ruleset (see Proposition 4 and Proposition 5).

In Theorem 2 we described a compact ruleset for a game with these P-positions – this time for *any* irrational $1 < \alpha < 2$. The meaning of "compact" here was that all the moves lie (asymptotically) on only 5 lines (see Figure 1(a)).

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